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CFT description of small objects in AdS

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Abstract: By the AdS/CFT correspondence, the expectation value of certain local operators in the CFT is given by the asymptotic value of supergravity fields. We show that these local expectation values contain a remarkable amount of information about small sources deep inside $AdS_p \times S^q$. In particular, they contain essentially all the multipole moments. More importantly, one can use them to determine the size of a spherical source. This is not a small effect: The size appears in an exponentially large contribution to the expectation values. This provides an easy way for the CFT to distinguish stars from black holes with the same mass, or to distinguish different "giant graviton" configurations.

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1. Introduction

One of the main open problems in the AdS/CFT correspondence [1, 2] is the following: What is the proper conformal field theory (CFT) description of objects in the bulk with size much less than the anti de Sitter (AdS) radius R? There is a hope that since physics on scales much less than R should be similar to that in flat space, this description would be an important step toward finding a holographic theory for asymptotically flat spacetimes. (For some work in this direction, see [3, 4].) We find evidence that this will not be the case. Special properties of the asymptotic $AdS_p \times S^q$ boundary conditions appear to be crucial in the CFT description of small objects in the bulk spacetime.

By the UV/IR connection [5, 6], one might expect that the description of small objects will require highly nonlocal operators [7, 8]. Surprisingly, this is not quite the case. Even though the expectation values of local operators in the gauge theory are sensitive only to the asymptotic value of supergravity fields, we will show that these leading order asymptotic fields contain a remarkable amount of information about small sources deep inside the spacetime.

In this respect, asymptotically $AdS_p \times S^q$ spacetimes are dramatically different from the more familiar asymptotically flat spacetimes. In the latter case, it is well known that the multipole moments of a source can be read-off from the asymptotic behavior of the field. But since higher multipole moments fall off faster, the leading asymptotic field depends only on the monopole moment — the total mass. In particular, there is no information about the size of the source. For a small source in $AdS_p \times S^q$, one might have expected that the situation would be very similar. After all, if the size r_0 of the source is much less than the radius of curvature R, there is an intermediate region $r_0 \ll r \ll R$ in which the spacetime is approximately flat and the information about the higher multipole moments will fall off more quickly. Nevertheless, we will show that for $r \gg R$, the situation is, in fact, very different. The $AdS_p \times S^q$ boundary conditions effectively "refocus" information about the source. It turns out that one can recover essentially all multipole moments of the source from the *leading* asymptotic behavior of the field. More importantly, one can recover the size of a spherical source from this asymptotic field.

We will focus mainly on the case of $AdS_5 \times S^5$ boundary conditions, which is described by an $\mathcal{N} = 4$ super Yang-Mills theory, although we will comment on other values of p and q where similar results are obtained. To illustrate the effect, we will consider the simplest possible case of a massless scalar field Φ with source s in $AdS_5 \times S^5$. This can be viewed as the dilaton. Perturbations of the metric or other supergravity fields should behave similarly.

It may seem surprising that one can say anything about the size of a source given the asymptotic value of the field. After all, given any static solution to the wave equation Φ in a neighborhood of infinity, one can continue it inside satisfying the field equation for a while. Then, at an arbitrary radius, one can take any smooth continuation of Φ to the origin and define $s \propto \nabla^2\Phi$. The result is a source, s, which produces the field Φ. So the size of the source is not in general fixed by the asymptotic field. However, in our case there is an additional constraint coming from symmetry. A small source which is spherically symmetric in ten dimensions will produce a field in $AdS_5 \times S^5$ which depends only on a radial distance χ in S^5 and a radial distance ρ in AdS_5. One could take this asymptotic field, cut it off at an arbitrary distance, extend it to the origin in any smooth way, and define a new source $s \propto \nabla^2\Phi$. But this source will in general be a function of both ρ and χ. It will not be approximately spherically symmetric. The requirement that the source be just a function of $r^2 \equiv \rho^2 + \chi^2$ places a strong constraint on how the asymptotic field can be matched onto a source. In particular, we will show that it fixes the size of the source.

The presence of the S^5 is crucial for obtaining this size information. Since a small source of size $r_0 \ll R$ breaks the SO(6) symmetry, there will be nonzero expectation values of operators involving $Z_l \equiv V_{j_1 \cdots j_l} X^{j_1} \cdots X^{j_l}$, where V is a symmetric traceless tensor of rank l, and X^j are the six scalars in the $\mathcal{N} = 4$ supersymmetric gauge theory. Given that the field in the intermediate region $r_0 < r < R$ is dominated by the monopole moment, one might have expected that, for certain local operators $\mathcal{O}$, $\langle \mathcal{O} Z_l \rangle$ would be universal and determined only by the mass of the source. However, this is not the case. We will see that the behavior of the expectation values $\langle \mathcal{O} Z_l \rangle$

changes dramatically when $l \sim (R/r_0)^2$.[1] They increase like l^2 for $l \ll (R/r_0)^2$, but grow exponentially with l for $l \gg (R/r_0)^2$! Thus it is easy to extract information about the size of the source.

Since we expect metric perturbations (which couple to the stress energy of all sources) to behave similarly to the scalar field Φ, there are several possible applications of this result. Trying to describe a small object in spacetime just from local operators in the gauge theory is analogous to astrophysicists trying to understand an exotic object in a distant galaxy by observing the radiation emitted. While it is relatively easy to get information about the mass of the object, it is usually difficult to directly measure its size. With $AdS_5 \times S^5$ boundary conditions, this difficulty is removed. So it is easy to determine that the source is a collapsed object rather than, e.g., a star. Another possible application concerns expanded brane configurations. It has recently been shown [9, 10, 11] that there are three different BPS configurations with the same mass and angular momentum as a graviton. In addition to the usual pointlike graviton, there is a three-brane expanded the AdS_5 directions, and another expanded in the S^5 directions. The question was raised in [11] as to how the CFT could distinguish these different configurations. Although we will focus on static sources, and not the moving branes required for these "giant gravitons", the results described here point toward a clear distinction between these three cases.

2. Recovering the source from the leading asymptotic field

The basic reason one can obtain information about small sources in $AdS_5 \times S^5$ from the leading order field at infinity is the following. Unlike asymptotically flat spacetimes, higher multipole moments of a field in AdS_5 do *not* fall off faster at infinity [12]. All modes of a field fall off at the same rate, so from the asymptotic value of the field at infinity, one can recover all the multipole moments of the source in the AdS_5 directions. The S^5 dependence of the source can be expanded in S^5 spherical harmonics (since they form a complete basis of functions). For sources which are a product of a function on S^5 and a function on AdS_5, if we could recover all the coefficients of this expansion, we could reconstruct not just the multipole moments but the entire source function in the S^5 directions. This is not quite possible, but from the asymptotic values of the corresponding Kaluza-Klein fields in AdS_5, we show below that even for general source functions, one can recover considerable information about the source in the S^5 directions, including its size.

Since the Kaluza-Klein modes will play a crucial role, we begin by deriving a simple property of solutions to massive wave equations in general static spacetimes.

[1] This is for spherical sources. For non-spherical sources, we will see that the change occurs at $l \sim (R/r_0)$.

Consider a massive wave equation with source s

$$\nabla^2\Phi - m^2\Phi = -ks \tag{2.1}$$

in a d-dimensional static spacetime with metric $ds^2 = -f^2dt^2 + g_{ij}dx^idx^j$. (The constant k is fixed by the normalization of Φ.) Static solutions to the wave equation satisfy[2]

$$D_i(fD^i\Phi) - m^2f\Phi = -ksf\,, \tag{2.2}$$

where D_i is the covariant derivative on a $t=$ constant surface Σ. The standard flat space identities relating integrals over the source to the asymptotic value of the field can easily be generalized to curved space. Let u be any nonsingular solution of the source-free equation

$$D_i(fD^iu) - m^2fu = 0\,. \tag{2.3}$$

Then we can multiply (2.2) by u and integrate over the spatial surface Σ to get

$$k\int_\Sigma usf = \oint_S \Phi fn^i\partial_iu - \oint_S ufn^i\partial_i\Phi\,, \tag{2.4}$$

where the integral on the right is over a sphere at infinity and n^i is its unit normal. This gives us a relation between an integral of the source and the asymptotic value of the field Φ.

We now consider some examples. For a massless field in flat spacetime, we can set $u_l = r^lY_l(\Omega_{d-2})$ where $Y_l(\Omega_{d-2})$ is a spherical harmonic on S^{d-2}. Since $f=1$, the left hand side of (2.4) is (k times one component of) the l^{th} multipole moment M_l. The right hand side will be finite only if the part of the field Φ with angular dependence Y_l falls off like $1/r^{d+l-3}$. Thus we recover the usual flat space result that higher multipole moments of the source are encoded in higher order terms in the asymptotic field.

Now consider AdS_5 with metric $ds^2 = -f^2(\rho)dt^2 + f(\rho)^{-2}d\rho^2 + \rho^2d\Omega_3^2$ with $f^2(\rho) = \rho^2/R^2 + 1$ where R is the radius of curvature. As noted in [12], all static solutions to the massless wave equation go to a constant plus $O(1/\rho^4)$ *regardless of their angular dependence.* (This will be reviewed below.) For normalizability, we require that the constant is zero, so

$$\Phi(\rho,\Omega_3) \to \frac{\phi(\Omega_3)}{\rho^4}\,. \tag{2.5}$$

Let $u_{\hat{l}}$ be the static solution that behaves like $\rho^{\hat{l}}Y_{\hat{l}}(\Omega_3)$ near the origin. If the size of the source is much smaller than R, then $f\approx 1$ over the extent of the source, and the left hand side of (2.4) is still a multipole moment $M_{\hat{l}}$. Since $u_{\hat{l}}$ now goes to a

[2] This can easily be seen by writing $\nabla^2\Phi = g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu\Phi)$ where $g = f^2\det g_{ij}$ is the determinant of the spacetime metric.

constant asymptotically, and $f \approx \rho/R$, $n^i\partial_i \approx (\rho/R)\partial/\partial\rho$ for large ρ, we see that $\phi(\Omega_3)$ contains all the information about the multipole moments of the source. More specifically, since $u_{\hat{l}}$ will grow like $\rho^{\hat{l}}$ until $\rho \sim R$, and then approach a constant asymptotically, we have $u_{\hat{l}} \sim R^{\hat{l}}$ for large ρ. Plugging this into (2.4) we see that the first term on the right vanishes and we have

$$\int_{S^3} \phi Y_{\hat{l}} \sim \frac{M_{\hat{l}}}{R^{\hat{l}-2}}\,. \tag{2.6}$$

So for R much larger than the size of the source ρ_0, the modes of the asymptotic field with $\hat{l} > 2$ are suppressed relative to the multipole moments, while for $\hat{l} < 2$ they are enhanced. (This can also be seen by dimensional analysis.) So even though all the multipole moments can be recovered from $\phi(\Omega_3)$, most of them make a very small contribution to the asymptotic field when $\rho_0 \ll R$. This is consistent with the fact that in the approximately flat region, $\rho_0 \ll \rho \ll R$, the modes of the field decay faster with increasing $\hat{l}$. However, the key point is that all the multipole moments can in principle be recovered from $\phi(\Omega_3)$.

We now turn to the case of interest, $AdS_5 \times S^5$. We will henceforth set $R = 1$ and measure all quantities in units of the AdS radius. The spacetime metric is thus

$$ds^2 = -(\rho^2+1)\,dt^2 + \frac{d\rho^2}{\rho^2+1} + \rho^2\,d\Omega_3^2 + d\Omega_5^2\,. \tag{2.7}$$

We again assume that the size of the source s is much smaller than the radius of curvature. Since spacetime is essentially flat near the source, we could take u near the source to be a spherical harmonic on S^8 times a power of the radius. Then, the right hand side of (2.4) would just be a (ten dimensional) multipole moment of the source. However, since the spacetime metric is a product of S^5 and AdS_5, it is much more convenient to expand all fields in spherical harmonics on S^5 and S^3.

The standard mode decomposition of a static scalar field in $AdS_5 \times S^5$ is

$$\Phi(\rho,\Omega_3,\Omega_5) = \sum_{L,\hat{L}} \frac{1}{\rho^{3/2}}\,\psi_{L,\hat{L}}(\rho)\,Y_{\hat{L}}(\Omega_3)\,Y_L(\Omega_5) \tag{2.8}$$

where $Y_{\hat{L}}(\Omega_3)$ and $Y_L(\Omega_5)$ are the spherical harmonics on the S^3 and S^5, respectively, and $L, \hat{L}$ label the different modes, e.g., $L = (l, \{m_i\})$. The spherical harmonics satisfy

$$\nabla^2_{\Omega_3} Y_{\hat{L}}(\Omega_3) = -\hat{l}(\hat{l}+2)\,Y_{\hat{L}}(\Omega_3) \tag{2.9}$$

and

$$\nabla^2_{\Omega_5} Y_L(\Omega_5) = -l(l+4)\,Y_L(\Omega_5)\,. \tag{2.10}$$

The source can be similarly decomposed:

$$s(\rho,\Omega_3,\Omega_5) = \sum_{L,\hat{L}} \frac{1}{\rho^{3/2}}\,\sigma_{L,\hat{L}}(\rho)\,Y_{\hat{L}}(\Omega_3)\,Y_L(\Omega_5)\,. \tag{2.11}$$

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The radial equation for $\psi_{L,\hat{L}}(\rho)$ resulting from (2.1) with $m = 0$ then becomes

$$\left(\rho^2 + 1\right) \psi''_{L,\hat{L}} + 2\rho\, \psi'_{L,\hat{L}} - V_{L,\hat{L}}\, \psi_{L,\hat{L}} = -k\sigma_{L,\hat{L}}\,, \tag{2.12}$$

where each term has an implicit ρ dependence and

$$V_{L,\hat{L}}(\rho) = \frac{3}{4\rho^2} + \frac{15}{4} + \frac{\hat{l}(\hat{l}+2)}{\rho^2} + l(l+4)\,. \tag{2.13}$$

Asymptotically, i.e. as $\rho \to \infty$, $V_{L,\hat{L}}(\rho) \to 15/4 + l(l+4)$ and $\sigma_{L,\hat{L}}(\rho) = 0$, so that (2.12) simplifies to

$$\rho^2\, \psi''_{L,\hat{L}} + 2\rho\, \psi'_{L,\hat{L}} - \left(\frac{15}{4} + l(l+4)\right) \psi_{L,\hat{L}} = 0\,. \tag{2.14}$$

This has the solution

$$\psi_{L,\hat{L}}(\rho) \sim \rho^{-(l+5/2)}\,, \tag{2.15}$$

which is the normalizable mode we wish to consider. For $l = 0$, this reduces to the standard result, $\Phi \sim \rho^{-4}$ quoted above. (The second solution, $\psi_{L,\hat{L}}(\rho) \sim \rho^{l+3/2}$, corresponds to the non-normalizable mode.) As is well known, from the pure AdS_5 (Kaluza-Klein reduced) picture, the higher spherical harmonics on the S^5 correspond to massive fields, with mass for the l^{th} mode given by $m_l^2 \equiv l(l+4)$. This confirms that the asymptotic falloff of a field (generated by a compact source) is given solely by the S^5 mode number l, and is independent of $\hat{l}$.

From (2.15) and (2.8), we are interested in the solution with asymptotic behavior

$$\Phi(\rho, \Omega_3, \Omega_5) \to \sum_{L,\hat{L}} \frac{M_{L,\hat{L}}}{\rho^{l+4}}\, Y_{\hat{L}}(\Omega_3)\, Y_L(\Omega_5)\,. \tag{2.16}$$

The coefficients $M_{L,\hat{L}}$ can be related to integrals of the source via (2.4). To use this, we need the exact solution to the source-free equation which is one at the origin. This is given by a hypergeometric function

$$R_{l,\hat{l}}(\rho) = \rho^{\hat{l}}\, F\left(\frac{\hat{l}-l}{2}, \frac{\hat{l}+l+4}{2}, \hat{l}+2; -\rho^2\right). \tag{2.17}$$

Setting $u = \sum_{L,\hat{L}} R_{l,\hat{l}}(\rho)\, Y^*_{\hat{L}}(\Omega_3)\, Y^*_L(\Omega_5)$ in (2.4) (where $*$ denotes complex conjugation) and using the asymptotic form of the hypergeometric function for large ρ, we obtain

$$M_{L,\hat{L}} = C_{l,\hat{l}} \int_\Sigma s(\rho, \Omega_3, \Omega_5)\, R_{l,\hat{l}}(\rho)\, Y^*_{\hat{L}}(\Omega_3)\, Y^*_L(\Omega_5)\, f d^9V\,, \tag{2.18}$$

where $f d^9V = \rho^3\, d\rho\, d\Omega_3\, d\Omega_5$, Σ denotes a constant-t slice of $AdS_5 \times S^5$, and

$$C_{l,\hat{l}} \equiv \frac{k}{2l+4}\, \frac{\Gamma(\frac{\hat{l}+l+4}{2})^2}{\Gamma(\hat{l}+2)\, \Gamma(l+2)}\,. \tag{2.19}$$

We are interested in the coefficients $M_{L,\hat{L}}$ since they are directly determined by the CFT expectation values. More precisely, for each L, the asymptotic fields $\sum_{\hat{L}} M_{L,\hat{L}} Y_{\hat{L}}(\Omega_3)$ are in a one to one correspondence with the expectation value of local operators in the gauge theory:

$$\langle \mathcal{O}(\Omega_3) Z_L(\Omega_3) \rangle \propto \sum_{\hat{L}} M_{L,\hat{L}} Y_{\hat{L}}(\Omega_3) \,. \tag{2.20}$$

The operators on the left are defined as follows. The label L specifies a particular spherical harmonic on S^5 which can be characterized by a symmetric traceless tensor $V_{j_1 \cdots j_l}$. The CFT operator Z_L is $Z_L = V_{j_1 \cdots j_l} X^{j_1} \cdots X^{j_l}$ where X^j are the six scalars in the CFT. The operator $\mathcal{O}$ depends on which supergravity field in the bulk one is considering. If Φ represents the dilaton, then $\mathcal{O} = F^2$. The total scaling dimension of $\mathcal{O} Z_L$ is thus $l + 4$.

How much information about the source is contained in the coefficients $M_{L,\hat{L}}$? We have already seen that for $L = 0$, corresponding to a massless field in AdS_5, the coefficients $M_{0,\hat{L}}$ give all the multipole moments of an effective source in the AdS_5 directions obtained by averaging the source over S^5. For $L \neq 0$, the coefficients are related to integrals of the source weighted by a spherical harmonic in the S^5 directions, and a solution to the massive wave equation in the AdS_5 directions. As long as the size of the source is much less than the mass of the field, i.e., small l, $R_{l,\hat{l}}(\rho) \approx \rho^{\hat{l}}$, and the coefficients again give standard multipole moments of an effective source in the AdS_5 directions.

We mentioned earlier that if one could obtain all the coefficients in the expansion of the source in modes on S^5, one could recover the entire source function in these directions. To see if this is possible, consider a source which is a product of a function on AdS_5 and a function on S^5, $s = s_1(\rho, \Omega_3) s_2(\Omega_5)$. It is clear from (2.18) that $M_{L,\hat{L}}$ is directly proportional to the coefficient $\sigma_{l,\{m_i\}}$ in the expansion of s_2 on S^5, and the remaining integral depends on l but is independent of $\{m_i\}$. Thus by taking ratios of $M_{L,\hat{L}}$ with the same l but different $\{m_i\}$, one can recover $\sigma_{l,\{m_i\}}$ up to one unknown constant for each l which can be taken to be $\sigma_{l,\{m_i=0\}}$. Roughly speaking, one can determine the non-SO(5) invariant part of the source on S^5 directly from ratios of $M_{L,\hat{L}}$, but one cannot determine the entire source function.

However, even for SO(5) invariant sources, one can recover a considerable amount of information. The most interesting case is spherically symmetric sources in ten dimensions. In this case, the only nonzero $M_{L,\hat{L}}$ have $\hat{L} = 0$ and $m_i = 0$, so they are labeled just by l. The behavior of these coefficients for increasing l is governed by a competition between the oscillating spherical harmonic on S^5, and the radial function $R_{l,0}(\rho)$, which grows *exponentially* with l for $\rho < 1$. This exponential growth is just a consequence of the fact that for small ρ, $R_{l,0}$ is a solution to a massive wave equation in nearly flat space with $m_l^2 = l(l+4)$. The solution which is finite at the origin grows like $e^{m_l\rho}/\rho^{3/2}$ for $\rho \gg 1/m_l$. This exponential growth can lead to exponentially large coefficients. Let us illustrate this with a simple flat space example.

Consider four dimensional Minkowski spacetime, $\mathcal{M}^4$. A conventional massive field satisfies (2.1) with $k = 4\pi$. Static solutions behave asymptotically like $\Phi = Ae^{-mr}/r$ for some constant A. Suppose the source s is a uniform density ball of radius r_0, i.e., $s = \Theta(r_0 - r)/\frac{4}{3}\pi r_0^3$. Then we can determine A by using (2.4). The solution to $(\partial_i\partial^i - m^2)u = 0$ with $u = 1$ at the origin is $u = \sinh mr/mr$. Substituting into (2.4) yields

$$A = \frac{3}{(mr_0)^3}[mr_0 \cosh mr_0 - \sinh mr_0]\,. \tag{2.21}$$

So for $mr_0 \gg 1$, $A \sim e^{mr_0}/(mr_0)^2$. Of course, outside the source, this exponentially large coefficient is more than compensated for by the exponentially small radial dependence, and the solution Φ is very small. Similar behavior occurs for small sources in AdS. But the point is that in the AdS/CFT correspondence, the radial dependence of the solution is scaled out, and the field theory expectation values are directly related to the (exponentially large) coefficients.

In some cases, this exponentially large contribution to the coefficient can be canceled by an oscillating contribution coming from compact extra dimensions. As a simple example, consider a massless field in $S^1 \times \mathcal{M}^4$. A spherical source in this space can be viewed as a periodic array of sources in $\mathcal{M}^5$. Since the spacetime is flat, we know that the asymptotic field can depend only on the monopole moment of the source (and the radius of the S^1). In particular, it is independent of the size of the source. On the other hand, if we expand both the source and the field in a Fourier series on S^1, the coefficient of the asymptotic field associated with each mode l is given by an expression similar to (2.18) involving the integral of exponentially growing and oscillating functions. Nevertheless, the integral is independent of the size of the source for each l. In the next section we will see that this precise cancellation does not extend to curved spacetimes such as $AdS_5 \times S^5$.

3. Size of spherical sources

We will now turn to one of the most interesting and important implications of our results, namely, that we can determine the size r_0 of a spherical object. (By "spherical" we mean invariant under SO(9) rotations.) First, we re-cap the formalism for a more general case of "bi-spherical" source, defined by $s(\rho, \Omega_3, \Omega_5) = s(\rho, \chi)$ where χ corresponds to the radial coordinate on the 5-sphere: $d\Omega_5^2 = d\chi^2 + \sin^2\chi\, d\Omega_4^2$. (Hence, a bi-spherical source generates a field which is spherically symmetric both on the S^3 of AdS_5 and on the S^4 of S^5, i.e. SO(9) is broken to SO(4) $\times$ SO(5).)

For a bi-spherically symmetric source, $s = s(\rho, \chi)$, only the modes with $\hat{l} = 0$ and $\{m_i = 0\}$ will be nontrivial, as pointed out in section 2. That means that we may consider just the spherical harmonics on S^5 which are functions of χ only. Eqs. (2.18) and (2.20) then simplify to

$$\langle \mathcal{O}\, Z_l \rangle \sim M_{l,\hat{0}} \sim \int_0^\pi \int_0^\infty s(\rho,\chi)\, R_l(\rho)\, Z_l(\chi)\, \rho^3\, \sin^4\chi\, d\rho\, d\chi\,, \tag{3.1}$$

where $R_l(\rho) \equiv R_{l,\hat{0}}(\rho)$ is given by (2.17) with $\hat{l}=0$, and $Z_l(\chi)$ is proportional to the $\{m_i = 0\}$ (real) spherical harmonic $Y_l(\chi)$ on S^5, which satisfies the equation[3]

$$Y_l''(\chi) + 4\frac{\cos\chi}{\sin\chi}\, Y_l'(\chi) + l(l+4)\, Y_l(\chi) = 0\,. \tag{3.2}$$

The corresponding solution, expressed in terms of hypergeometric functions, is $Y_l(\chi) = \gamma_l\, F(-l/2, l/2+2, 1/2; \cos^2\chi)$ for even l, and $Y_l(\chi) = \gamma_l\, \cos\chi\, F((1-l)/2, (5+l)/2, 3/2; \cos^2\chi)$ for odd l, where γ_l is a constant fixed by orthonormality of the Y_l's.

Let us now make a few comments on normalization, implicit in the "$\sim$" sign of (3.1). First, linearity of the Poisson equation ensures that the $\langle \mathcal{O}\, Z_l\rangle$'s of a given source will be directly proportional to the source's mass M. Thus, for simplicity, we can confine our discussion to unit mass sources, or equivalently, consider the expectation values with the mass scaled out. Second, since $\langle \mathcal{O}\, Z_l\rangle$ have different scaling dimensions for different l, we cannot compare them directly; we can only meaningfully compare operators with the same dimension. This suggests dividing $\langle \mathcal{O}\, Z_l\rangle$ of the given source by $\langle \mathcal{O}\, Z_l\rangle$ of a convenient "standard reference source" such as unit mass δ-function source. Thus, we define the normalized expectation values

$$\langle \mathcal{O}\, Z_l\rangle_n \equiv \frac{1}{M}\, \frac{\langle \mathcal{O}\, Z_l\rangle_s}{\langle \mathcal{O}\, Z_l\rangle_\delta} \tag{3.3}$$

We can treat $\langle \mathcal{O}\, Z_l\rangle_n$ as pure numbers (which depend on the mode l and the source function s). As a by-product, the l-dependent coefficients $C_{l,\hat{0}}$ (2.19) coming into the definition (2.18) of $M_{l,\hat{0}}$, as well as the normalization γ_l of the spherical harmonics, cancel out.

Evaluating $\langle \mathcal{O}\, Z_l\rangle$ explicitly for a unit-mass δ-function source, $s_\delta(\rho,\chi) = \frac{\delta(\rho)}{2\pi^2\rho^3} \times \frac{\delta(\chi)}{(8\pi^2/3)\,\sin^4\chi}$, then leads to the following formula:

$$\begin{aligned}\langle \mathcal{O}\, Z_l\rangle_n = \frac{1}{M}\frac{16\pi^4}{3} \int_0^\pi \int_0^\infty & s(\rho,\chi)\, F\left(-\frac{l}{2}, \frac{l}{2}+2, 2; -\rho^2\right) \times \\ & \times \frac{F\left(-\frac{l}{2}, \frac{l}{2}+2, \frac{1}{2}; \cos^2\chi\right)}{F\left(-\frac{l}{2}, \frac{l}{2}+2, \frac{1}{2}; 1\right)}\, \rho^3\, \sin^4\chi\, d\rho\, d\chi \end{aligned} \tag{3.4}$$

[3]There is a slight abuse of notation here. We have previously defined Z_l to be the gauge theory operator $V_{j_1\cdots j_l} X^{j_1}\cdots X^{j_l}$ where V is symmetric and traceless. The spherical harmonic $Z_l(\chi)$ denotes (up to a normalization constant) the function on S^5 obtained by taking the same tensor V and contracting all indices with a unit vector in R^6.

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for even l, and the analogous expression for odd l. From the form of (3.4) it is convenient to normalize $Z_l(\chi)$ such that $Z_l(\chi = 0) \equiv 1 \ \forall l$. In other words

$$Z_l(\chi) \equiv \frac{F\left(-\frac{l}{2}, \frac{l}{2}+2, \frac{1}{2}; \cos^2\chi\right)}{F\left(-\frac{l}{2}, \frac{l}{2}+2, \frac{1}{2}; 1\right)} \tag{3.5}$$

for even l, and similarly for odd l. As an immediate consequence of (3.4), for $l = 0$, $\langle \mathcal{O}\, Z_0 \rangle_n = \langle \mathcal{O} \rangle_n = 1$ for all sources.

We can use (3.4) to evaluate $\langle \mathcal{O}\, Z_l \rangle_n$ numerically, in principle for arbitrary l. However, for wide range of l's, we can obtain the answer much more easily and efficiently by using series approximations, which are good for small enough values of l. We first describe this in detail and present several results for small l. The most important result is the fact that for any $l > 0$, we can extract the size of a uniform density, spherically symmetric source from $\langle \mathcal{O}\, Z_l \rangle_n$. We then turn to large l, where we consider $\langle \mathcal{O}\, Z_l \rangle_n$ for the same sources. We will find that the corresponding results are even more striking.

3.1 Small l

For a small source (of size $r_0 \ll 1$ in AdS units) localized on the north pole of S^5, the integration in (3.4) only runs over $\rho \ll 1$ and $\chi \ll 1$, so we can approximate the functions appearing in (3.4) by Taylor series in ρ and χ. In particular,

$$R_l(\rho) = F\left(-\frac{l}{2}, \frac{l}{2}+2, 2; -\rho^2\right) \approx 1 + \frac{l(l+4)}{8}\rho^2 + \frac{(l-2)l(l+4)(l+6)}{192}\rho^4 + \cdots \tag{3.6}$$

and

$$Z_l(\chi) \approx 1 - \frac{l(l+4)}{10}\chi^2 + \frac{l(l+4)(3l^2+12l-8)}{840}\chi^4 + \cdots \tag{3.7}$$

(for both even and odd l). For a source with maximal extent r_0, $\rho \leq r_0 \ll 1$ and $\chi \leq r_0 \ll 1$, so that the above series expansion is a good approximation provided that $l\, r_0 \ll 1$, i.e. $l \ll 1/r_0$. The final result, $\langle \mathcal{O}\, Z_l \rangle_n$, will then be expressed as an expansion in powers of r_0.

A uniform spherical source of mass M and radius r_0 in ten dimensional flat spacetime would be described by $s(\rho, \chi) = \frac{M}{V_9 r_0^9}\, \Theta(r_0^2 - (\rho^2 + \chi^2))$ where Θ is the Heavyside function and $V_9 = 2^5\pi^4/9!!$ denotes the volume of a unit ball in R^9. (The denominator normalizes the mass, so that $\int s\, d^9V = \frac{16\pi^4}{3} \int_0^\pi \int_0^\infty s(\rho,\chi)\, \rho^3\, \chi^4\, d\rho\, d\chi = M$, independently of r_0.) In exactly flat space, we of course would not be able to extract the size r_0 from the asymptotic value of the field. We will now see how the situation differs in $AdS_5 \times S^5$.

The $AdS_5 \times S^5$ spacetime deviates from the flat $\mathcal{M}^{10}$ spacetime at quadratic order in ρ, so we might expect quadratic modifications to the "natural" source. In fact, there are two distinct modifications. First, since the mass M, given by the integral of the source, will depend on the measure: $M = \frac{16\pi^4}{3} \int_0^\pi \int_0^\infty s(\rho,\chi)\, \rho^3\, \sin^4\chi\, d\rho\, d\chi$, the volume normalization V_9 will receive r_0-dependent corrections, $V_9 \to W_9 = V_9(1 + O(r_0^2))$. Second, a spherical object should have the same extent in the AdS and the sphere directions in terms of the proper distance, rather than the coordinate distance. Specifically, the proper distance in the AdS and the sphere directions is given respectively by

$$\hat{\rho} = \int^{\rho} \frac{d\tilde{\rho}}{\sqrt{\tilde{\rho}^2 + 1}} = \sinh^{-1} \rho \,, \qquad \hat{\chi} = \chi \,. \tag{3.8}$$

Hence, a truly spherically symmetric (uniform density) source should be written as a function of $r^2 \equiv \hat{\rho}^2 + \hat{\chi}^2$:

$$\begin{aligned} s(r) &= \frac{M}{W_9 r_0^9} \Theta(r_0 - r) = \frac{M}{W_9 r_0^9} \Theta\left(r_0^2 - (\sinh^{-1}\rho)^2 - \chi^2\right) \\ &\approx \frac{M}{V_9 r_0^9 (1 + \frac{2}{33} r_0^2 + \frac{49}{2145} r_0^4 + \cdots)} \Theta\left(r_0^2 - \rho^2\left(1 - \frac{1}{3}\rho^2 + \cdots\right) - \chi^2\right) \end{aligned} \tag{3.9}$$

which will give $\langle \mathcal{O}\, Z_l \rangle_n$ to $O(r_0^6)$. (The latter modification will only become relevant at the quartic order.)

We now evaluate $\langle \mathcal{O}\, Z_l \rangle_n$ explicitly, up to $O(r_0^6)$, for general l (which is small enough so that the series expansion in r_0 is still useful). Using the approximations (3.9), (3.6), and (3.7), we find that

$$\langle \mathcal{O}\, Z_l \rangle_n = 1 + \frac{5l(l+4)}{858} r_0^4 + O(r_0^6) \,. \tag{3.10}$$

Explicitly, if X_1 corresponds to the direction of the north pole ($\chi = 0$) on S^5, the first few expectation values are:

$$\begin{aligned} \langle \mathcal{O}\, X_1 \rangle_n &= 1 + \frac{25}{858} r_0^4 + O(r_0^6) \\ \left\langle \mathcal{O} \left(\frac{6}{5} X_1^2 - \frac{1}{5} \sum_i X_i X^i \right) \right\rangle_n &= 1 + \frac{10}{143} r_0^4 + O(r_0^6) \end{aligned} \tag{3.11}$$

and so on.

From (3.10), we discover that the size dependence in $\langle \mathcal{O}\, Z_l \rangle_n$ first appears at the quartic order in r_0, and the effect grows only quadratically with l. This is illustrated in figure 1, where $\langle \mathcal{O}\, Z_l \rangle_n$ is plotted as a function of the mode number l, for a particular small value of the source radius ($r_0 = 0.1$). The dashed curve corresponds to the quartic approximation (3.10), while the solid curve shows the true behavior. Before discussing the size dependence itself, let us note two interesting facts about the approximation (3.10). As is apparent from figure 1, the quartic approximation is valid for a much larger range of l than one would naively expect, i.e. even for $l \gg 1/r_0$ (= 10 in this case). In fact, for general r_0, the quartic order approximation

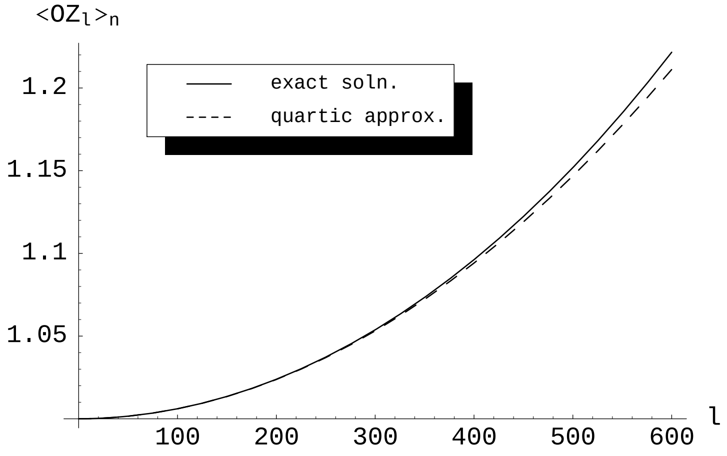


Figure 1: $\langle \mathcal{O}\, Z_l \rangle_n$ from (3.4) (solid curve), and the quartic approximation (dashed curve), for uniform density, spherically symmetric source with radius $r_0 = 0.1$.

is good to within a fraction of a percent all the way up to $l \sim O(1/r_0^2)$. Also, the approximation is smaller than the exact behavior, which seems to be the case for higher order approximations as well.

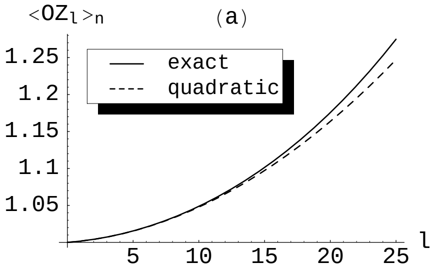
As we have seen from (3.10), we may extract the size of the (uniform density, spherically symmetric) source only at $O(r_0^4)$. At the first glance, this may seem somewhat surprising, since one might have expected that the quadratic deviations of the metric from flat spacetime should translate into quadratic effect of the size, i.e. $\langle \mathcal{O}\, Z_l \rangle_n = 1 + O(r_0^2)$ rather than $\langle \mathcal{O}\, Z_l \rangle_n = 1 + O(r_0^4)$. We may also gain similar expectations from the pure AdS case: The l-dependence of the leading asymptotic fall-off of a massive field (with mass $m_l \equiv l(l+4)$) and spherically symmetric (in AdS_5) source of size ρ_0 is proportional to $1 + \frac{l(l+4)}{12}\rho_0^2 + 0(\rho_0^4)$. The analogous set-up for the full $AdS_5 \times S^5$ case would involve a separable source, such as $s(\rho,\chi) \propto \Theta(\rho_0 - \rho)\, \Theta(\chi_0 - \chi)$, for which we indeed find

$$\langle \mathcal{O}\, Z_l \rangle_n = \left(1 + \frac{l(l+4)}{12}\rho_0^2 + 0(\rho_0^4)\right) \left(1 - \frac{l(l+4)}{14}\chi_0^2 + 0(\chi_0^4)\right). \tag{3.12}$$

Even when the source is no longer separable, we would still expect the size information to appear at the quadratic level, unless there is a very special cancellation.

Such special cancellation at the $O(r_0^2)$ does indeed occur for spherically symmetric sources. In fact, for *any spherically symmetric* source, the $O(r_0^2)$ contribution in $\langle O Z_l \rangle$ will vanish. Although we will show this explicitly momentarily, it is perhaps more instructive to first consider an a-spherical (but still bi-spherical) source. The simplest such case is the uniform density ellipsoidal source: $s(\rho,\chi) = \frac{M}{W_9 r_0^9}\, \Theta(r_0^2 - (\hat{\rho}^2/a^2 + \hat{\chi}^2/b^2))$, where $W_9 = V_9\, a^4 b^5\, \left(1 + \frac{2}{33}(6a^2 - 5b^2) r_0^2 + 0(r_0^4)\right)$, and $\hat{\rho}$, $\hat{\chi}$ are given by (3.8). Then we obtain the following result:

$$\langle \mathcal{O}\, Z_l \rangle_n = 1 + \frac{l(l+4)}{22}(a^2 - b^2) r_0^2 + O(r_0^4) \tag{3.13}$$



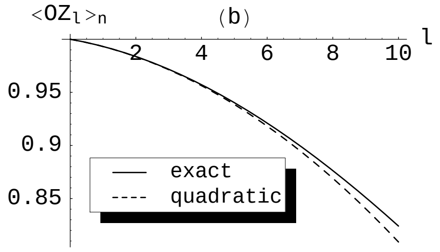


Figure 2: $\langle \mathcal{O}\, Z_l \rangle_n$ from (3.4) (solid curve), and the quadratic approximation (3.13) (dashed curve), for $r_0 = 0.1$, for an ellipsoidal, uniform density source extended to r_0 in the AdS directions and to (a) $r_0/2$ and (b) $2r_0$ in the sphere directions.

Thus we see that indeed, for the spherically symmetric case, $a = b$, the $O(r_0^2)$ contribution vanishes.[4]

On the other hand, for any non-spherical ($a \neq b$) uniform density source, the size is visible already at the quadratic level, from $(a^2 - b^2)\, r_0^2$. We show this in figure 2, where we plot $\langle \mathcal{O}\, Z_l \rangle_n$ (again as a function of l) for $r_0 = 0.1$, for source extended to r_0 in the AdS direction and (a) squashed to $r_0/2$ along the sphere ($a = 1, b = 1/2$) and (b) stretched to $2r_0$ along the sphere ($a = 1, b = 2$). Comparison with figure 1 shows that now we can see the size effect much earlier in l; fact, the relevant scale in l seems to be given by $1/r_0$, as we had originally expected, rather than by $1/r_0^2$, as we discovered for the spherically symmetric case, $a = b$.

Figure 2 also demonstrates another important characteristic of the $\langle \mathcal{O}\, Z_l \rangle_n$'s for a-spherical sources: they grow if the source is more extended in the AdS directions, whereas they decrease if the source has a bigger extent along the sphere. This is consistent with our naive expectations that in the former case, the exponentially growing radial solution in AdS "wins out" over the oscillatory contribution from the spherical harmonic, whereas in the latter case this is reversed: the oscillations damp out the growing mode. For the spherical case, then, these two effects balance out (though unlike in the flat $S^1 \times \mathcal{M}^4$ example of section 2, in $AdS_5 \times S^5$, they do not cancel completely).

One might wonder whether this special cancellation for spherically symmetric sources ($a = b$) had anything to do with the source density profile. We will now show that quite generally, for *any* spherically symmetric source, the quadratic terms will cancel. Although this is intuitively obvious by considering appropriate superposition of uniform density sources of various sizes, we can see a simple proof by changing

[4]From the quartic term, which has the r_0^4 coefficient

$$\frac{l(l+4)}{37752}\left[33\,(a^2-b^2)^2\, l^2 + 132\,(a^2-b^2)^2\, l + 4\,(9a^4 + 44a^2b^2 + 2b^4)\right],$$

we see that the $O(r_0^4)$ terms which grow faster than quadratically with l also vanish for $a = b$.

variables to make the spherical symmetry more explicit:

$$\hat{\rho} = r\,\sin\theta\,, \qquad \hat{\chi} = r\cos\theta \tag{3.14}$$

where $0 \le \theta \le \pi/2$. The source will now be a function of r only, i.e. $s = s(r)$, so that the integration over θ is independent of the source. Then, to quadratic order,

$$\langle \mathcal{O}\, Z_l \rangle_n \propto \int_0^\infty s(r)\, r^8 \int_0^{\pi/2} \sin^3\theta\, \cos^4\theta\, \left(1 + h_l(\theta)\, r^2 + O(r^4)\right)\, d\theta\, dr\,, \tag{3.15}$$

where $h_l(\theta) \equiv -\frac{3}{2}\cos^2\theta - \frac{l(l+4)}{10}\cos^2\theta + \frac{l(l+4)}{8}\sin^2\theta$. The three terms in h_l arise from the integration measure on the 5-sphere, the spherical harmonic $Z_l(\chi)$ expansion, and the radial solution $R_l(\rho)$ expansion, respectively.) Since the l-dependent part of $\int_0^{\pi/2}\sin^3\theta\,\cos^4\theta\, h_l(\theta)\, d\theta$ vanishes (and the l-independent part cancels the mass normalization), the quadratic term disappears, leaving us with $\langle \mathcal{O}\, Z_l\rangle_n \propto \int_0^\infty s(r)\, r^8\, (1 + O(r^4))\, dr$. For a source with an extent r_0, this yields $\langle \mathcal{O}\, Z_l\rangle_n = 1 + O(r_0^4)$. One might then worry that there will be even higher order cancellations; however, performing the above calculation to the next order yields a nonzero r_0^4 coefficient, consistent with our previous calculations.

Thus, we have seen that with sufficient precision, we can extract the size of the object from the $\langle \mathcal{O}\, Z_l\rangle_n$ values. (The fact that for spherical objects, this effect comes only at the quartic, rather than quadratic, order, just means that we would need greater precision.) In all cases, however, the magnitude of the size-dependent perturbation of the normalized expectation values $\langle \mathcal{O}\, Z_l\rangle_n$ increases with increasing l. Before considering the large l regime, where this effect is quite pronounced, let us continue examining how much information about the size can one obtain from $\langle \mathcal{O}\, Z_l\rangle_n$ in the small l regime.

For uniform density "spherical" sources, we have demonstrated that we can extract the size from the $\langle \mathcal{O}\, Z_l\rangle_n$ values. However, extracting the size in this way requires a knowledge of the source's density profile, apart from spherical symmetry. In general, we may not wish to rely on such detailed information (since even for a small star in $AdS_5 \times S^5$, we don't know its density profile), and obtaining drastically different coefficients of the r_0 powers for different profiles would undermine our method's usefulness.

Hence, to see if this in fact happens, we now consider a spherical source with several different density profiles; in particular,

$$s(r) \propto (r_0^2 - r^2)^\nu\, \Theta(r_0 - r) \tag{3.16}$$

where we choose the exponent $\nu = 0, 1, 2,$ and 4. ($\nu = 0$ corresponds to the previously discussed uniform density case). These are plotted in figure 3*a*, with ν increasing from top curve to bottom curve (i.e. the long-dash curve corresponds to the $\nu = 4$ case). The corresponding $\langle \mathcal{O}\, Z_l\rangle_n$'s (plotted as functions of l) for $r_0 = 0.1$ are shown

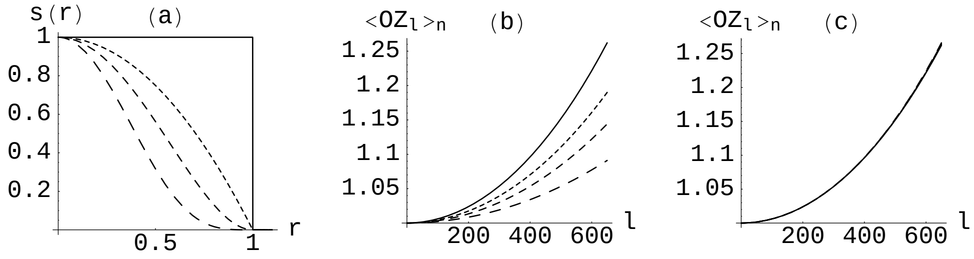


Figure 3: For the source density profiles (3.16) with $\nu = 0$ (solid curves), $\nu = 1$ (shortest-dash curves), $\nu = 2$ (longer-dash curves), and $\nu = 4$ (longest-dash curves), (*a*) the density profiles, (*b*) $\langle \mathcal{O}\, Z_l \rangle_n$'s from (3.4) for $r_0 = 0.1$, and (*c*) $\langle \mathcal{O}\, Z_l \rangle_n$'s from (3.4) for $r_0 = 0.1$, 0.108, 0.116, and 0.129, respectively.

in figure 3*b*. (The solid curve in figure 3*b* corresponds to the solid curve in figure 1.) The $\langle \mathcal{O}\, Z_l \rangle_n$'s behave just as we might expect: the sources with sharper profiles look effectively smaller, which is reflected in slower rise of the $\langle \mathcal{O}\, Z_l \rangle_n$'s with l.

The quartic approximations for the other curves are given by

$$
\begin{aligned}
\langle \mathcal{O}\, Z_l \rangle_n &= 1 + \frac{l(l+4)}{234} r_0^4 + 0(r_0^6) \qquad \text{for } \nu = 1 \\
\langle \mathcal{O}\, Z_l \rangle_n &= 1 + \frac{l(l+4)}{306} r_0^4 + 0(r_0^6) \qquad \text{for } \nu = 2 \\
\langle \mathcal{O}\, Z_l \rangle_n &= 1 + \frac{5l(l+4)}{2394} r_0^4 + 0(r_0^6) \qquad \text{for } \nu = 4\,.
\end{aligned} \tag{3.17}
$$

Comparing (3.10) with (3.17), we find that the sizes r_0 required to produce the same quartic behavior of $\langle \mathcal{O}\, Z_l \rangle_n$ only differ by factors of ~ 1.08, 1.16, and 1.29, for $\nu = 1, 2$, and 4, respectively. Thus, if we only had precision up to the quartic order, we could expect to determine the size of the source (for reasonable density profiles) only up to factors of order unity. It turns out that even for the exact solutions for $l < O(1/r_0^2)$, the $\langle \mathcal{O}\, Z_l \rangle_n$ curves corresponding to different density profiles have very similar shapes. This is shown in figure 3*c*, where the four $\langle \mathcal{O}\, Z_l \rangle_n$'s are plotted as in figure 3*b*, except that the total size of each source is scaled as above, so as to produce the identical quartic behavior. Note that these exact curves coincide with each other, even though they differ from their (common) quartic approximation, as seen in figure 1.

These results imply that without knowing the source's density profile, we could extract its size only up to factors of order unity. While this would be sufficient for the applications discussed in the Introduction, we will see that we can actually do better, by considering the large l regime.

3.2 Large l

We now turn to large l. In particular, we wish to consider the regime where the quartic approximations considered above are no longer useful. One might have ex-

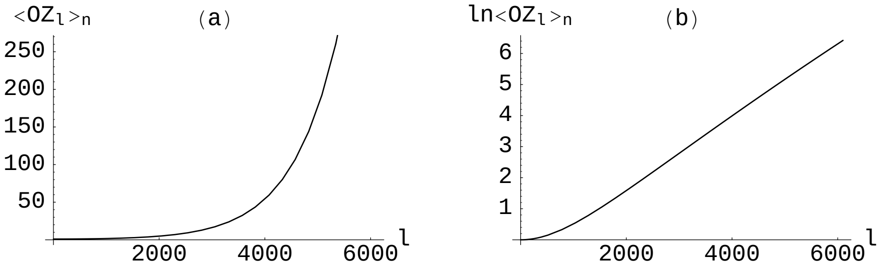


Figure 4: Exponential growth of $\langle \mathcal{O}\, Z_l \rangle_n$ at large l for uniform density spherically symmetric source of size $r_0 = 0.1$: (*a*) $\langle \mathcal{O}\, Z_l \rangle_n$ and (*b*) $\ln\langle \mathcal{O}\, Z_l \rangle_n$.

pected that this would translate into $l \gg 1/r_0$, since the series expansions of (3.6) and (3.7) are effectively series in $(l\rho)$ and $(l\chi)$, respectively, and hence $\langle \mathcal{O}\, Z_l \rangle_n$ should be a series in (lr_0). While this intuition is correct for generic sources (as confirmed explicitly for the ellipsoidal source cases), for spherical sources, it turns out that the special cancellations lead to effective series in (lr_0^2) (as was foreshadowed in the previous subsection). Hence, for the case of spherical sources, "large l" regime will mean $l \gg 1/r_0^2$.[5]

We could simply evaluate (3.4) numerically, but it still proves more efficient to use the series expansion to very high order (such as to $O(r_0^{40})$). This method is much faster, and we have good control over our accuracy by examining the relative contribution of the individual terms. The analogous result as that presented in figure 1 (i.e. the continuation of the solid curve) is shown in figure 4*a*, where we plot $\langle \mathcal{O}\, Z_l \rangle_n$ for uniform density spherical source of size $r_0 = 0.1$ up to $l \sim 6000$ (where the value of $\langle \mathcal{O}\, Z_l \rangle_n$ still has $< 1\%$ uncertainty). We now see that $\langle \mathcal{O}\, Z_l \rangle_n$ rises much faster than quadratically with l; it in fact rises exponentially! This is verified in figure 4*b*, where the logarithm of $\langle \mathcal{O}\, Z_l \rangle_n$ is plotted as a function of l, and asymptotically approaches a straight line.

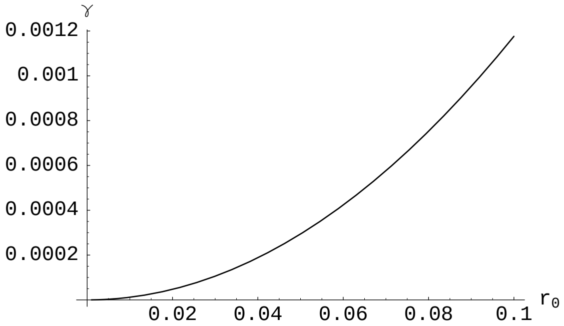


Figure 5: Exponent γ of $\langle \mathcal{O}\, Z_l \rangle_n \sim e^{\gamma l}$ at large l for uniform density spherically symmetric source of size r_0.

The natural question to ask at this point is, how does this exponential depend on the size r_0? In particular, does $\langle \mathcal{O}\, Z_l \rangle_n \sim e^{r_0 l}$ as one might expect from the $R_l \sim e^{l\rho}/(l\rho)^{3/2}$ exponential behavior, or does the cancellation seen for small l continue into the large l regime? The answer is shown in figure 5, where (for $\langle \mathcal{O}\, Z_l \rangle_n = c\, e^{\gamma l}$)

[5] In the AdS/CFT correspondence at finite N, there is an upper limit to l coming from the stringy exclusion principle [13] given by $l < N$. However the AdS radius is proportional to $N^{1/4}$ in Planck units, so modes with $l \sim N$ have wavelengths much shorter than the Planck scale. Since our sources are much larger than the Planck scale, our large l regime is $1/r_0^2 \ll l \ll N$.

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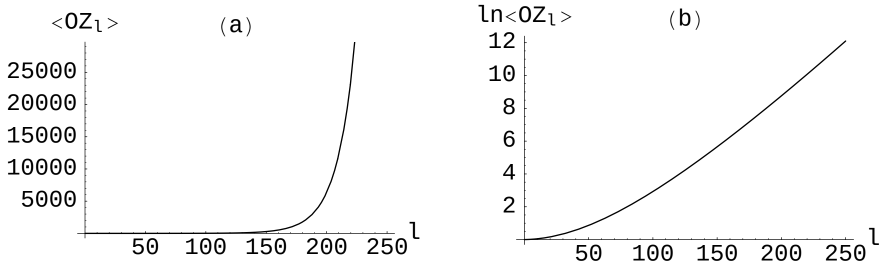

Figure 6: Exponential growth of $\langle \mathcal{O}\, Z_l \rangle_n$ at large l for uniform density ellipsoidal source source of size $r_0 = 0.1$ in AdS directions and size $r_0/2$ in sphere directions: (a) $\langle \mathcal{O}\, Z_l \rangle_n$ and (b) $\ln\langle \mathcal{O}\, Z_l \rangle_n$.

the exponent γ is plotted as a function of r_0. Actually, γ is approximated very well by $\gamma(r_0) = 0.118\, r_0^2$, with the two curves indistinguishable in figure 5. Numerically, we find that for large l,

$$\langle \mathcal{O}\, Z_l \rangle_n \approx 0.49\, e^{0.118\, l\, r_0^2} \tag{3.18}$$

Hence, the scaling of $\langle \mathcal{O}\, Z_l \rangle_n$ with r_0 is indeed slower than one might have naively expected (i.e. the exponent varies only as $0.118\, r_0^2$ rather than r_0). This result is nonetheless quite remarkable, since it says that in the large l regime, the normalized gauge theory expectation values are exponentially sensitive to r_0^2.

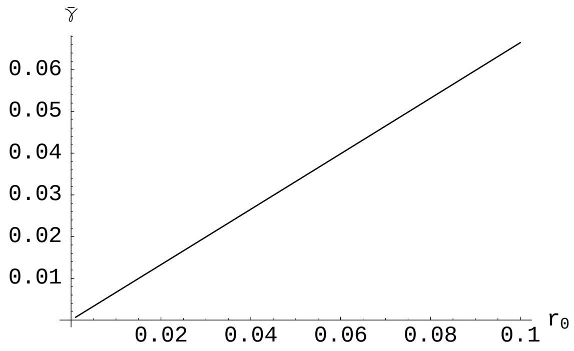

Figure 7: Exponent $\bar{\gamma}$ of $\langle \mathcal{O}\, Z_l \rangle_n \sim e^{\bar{\gamma}\, l}$ at large l for uniform density ellipsoidal source of size r_0 in AdS directions and size $r_0/2$ in sphere directions.

The preceding results suggest that we can obtain even faster growing $\langle \mathcal{O}\, Z_l \rangle_n$'s, if we consider a-spherical sources. Let us therefore use the ellipsoidal sources of the previous section (cf. figure 2). First, consider the source squashed in the sphere directions, or equivalently, stretched in the AdS directions. The corresponding $\langle \mathcal{O}\, Z_l \rangle_n$ is plotted (as a function of l) in figure 6a. From the values of $\langle \mathcal{O}\, Z_l \rangle_n$, we confirm that these indeed grow much faster with l than for the non-squashed case, figure 4. Not too surprisingly, the behavior again approaches an exponential, as seen from the corresponding plot of the logarithm of $\langle \mathcal{O}\, Z_l \rangle_n$, figure 6$b$.

As previously, we can analyze the r_0 dependence of the exponent, $\bar{\gamma}$ in $\langle \mathcal{O}\, Z_l \rangle_n = \bar{c}\, e^{\bar{\gamma}\, l}$. This is plotted in figure 7 (which is exactly analogous to figure 5). Unlike for the spherically symmetric case, the exponent now does vary linearly with r_0, rather than quadratically. (This gives rise to the faster growth of these expectation values.) More specifically, we find that the behavior of $\langle \mathcal{O}\, Z_l \rangle_n$ for large l may be

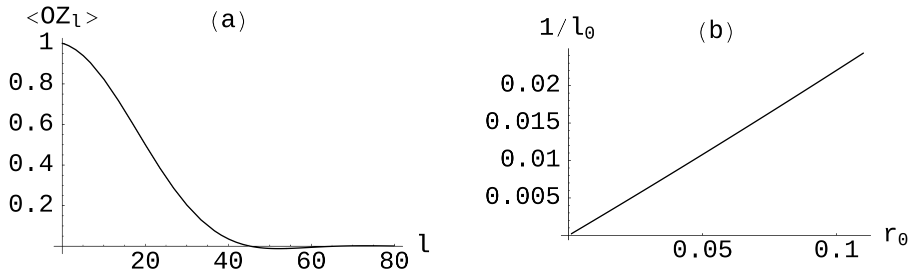

Figure 8: For uniform density ellipsoidal source source of size r_0 in AdS directions and size $2r_0$ in sphere directions: (a) $\langle \mathcal{O}\, Z_l \rangle_n$ from (3.4) for $r_0 = 0.1$, (b) inverse l at which $\langle \mathcal{O}\, Z_l \rangle_n$ first crosses zero for given r_0.

approximated by

$$\langle \mathcal{O}\, Z_l \rangle_n \approx 0.01\, e^{0.67\, l\, r_0} \tag{3.19}$$

Let us now briefly consider the source extended in the sphere directions. The continuation of $\langle \mathcal{O}\, Z_l \rangle_n$ from figure 2b to larger values of l (namely, to $l \sim 80$) is given in figure 8a. The initial drop of $\langle \mathcal{O}\, Z_l \rangle_n$ is followed by damped oscillations around $\langle \mathcal{O}\, Z_l \rangle_n = 0$. While this is perhaps not as glamorous as exponentially growing $\langle \mathcal{O}\, Z_l \rangle_n$'s, it is certainly a significant signature of the size of the source. Furthermore, this case illustrates that $\langle \mathcal{O}\, Z_l \rangle_n$ can actually change sign for appropriate l. For completeness, we can extract the characteristic r_0 dependence by considering, for instance, $l \equiv l_0$ defined as the l for which $\langle \mathcal{O}\, Z_l \rangle_n$ first vanishes. This should be inversely proportional to r_0, since for smaller sources, $\langle \mathcal{O}\, Z_l \rangle_n$ should fall off slower with l and therefore cross zero later. Plotting $1/l_0$ for different r_0, as shown in figure 8b, indeed verifies this to be the case.

The above results, summarized in figures 5, 7, and 8b, confirm our expectation that the characteristic scale for l of $\langle \mathcal{O}\, Z_l \rangle_n$ is $O(1/r_0^2)$ for spherically symmetric sources but only $O(1/r_0)$ for ellipsoidal sources. Thus, ellipsoidal sources in $AdS_5 \times S^5$ (extended in the AdS) would have an even more spectacular signature in the gauge theory. Although we presented the results for rather large a-sphericities (where the extent of the source in the AdS and sphere directions differed by a factor of 2), we find that the same basic behavior holds for any ellipsoidal source with non-zero a-sphericity. (This is a direct extension of the behavior summarized in (3.13).)

We have considered the behavior of $\langle \mathcal{O}\, Z_l \rangle_n$ for ellipsoidal sources mainly to contrast this with the behavior of the (presumably more physically relevant) spherically symmetric sources. However, one can ask, how physically relevant *are* the ellipsoidal sources (perhaps with very tiny a-sphericities)? In other words, do we expect, say, "stars" in $AdS_5 \times S^5$ to be exactly spherically symmetric? We cannot answer this question without knowing the detailed equation of state, etc. However, we note that since the tidal forces will be different in the AdS and the sphere directions, spherically symmetric density profile of the star would preclude spherically symmetric pressure

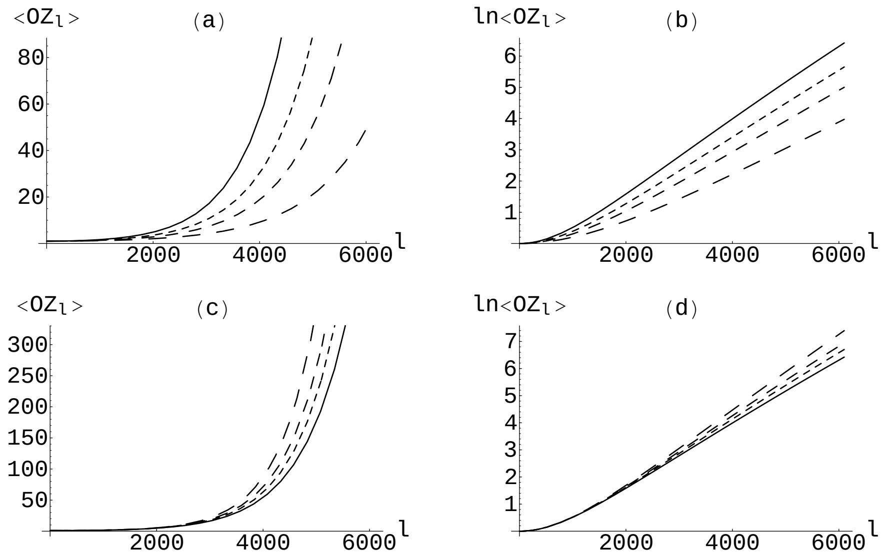

Figure 9: For the source density profiles (3.16) with $\nu = 0$ (solid curves), $\nu = 1$ (shortest-dash curves), $\nu = 2$ (longer-dash curves), and $\nu = 4$ (longest-dash curves), *(a)* $\langle \mathcal{O}\, Z_l \rangle_n$ and *(b)* $\ln\langle \mathcal{O}\, Z_l \rangle_n$ for $r_0 = 0.1$; *(c)* $\langle \mathcal{O}\, Z_l \rangle_n$ and *(d)* $\ln\langle \mathcal{O}\, Z_l \rangle_n$ for $r_0 = 0.1$, 0.108, 0.116, and 0.129, respectively.

profile. Conversely, for spherically symmetric pressure, the source can't be expressed as a function of r. Such a source might then look more like an ellipsoid (presumably squashed in the ρ direction due to the AdS potential).

Finally, to complete the large l discussion in parallel with the small l analysis above, let us now consider spherical sources with various density profiles. We have seen that in the small l regime, we could not distinguish certain sources with different profiles and correspondingly different sizes from each other. Since at large l, $\langle \mathcal{O}\, Z_l \rangle_n$ becomes exponentially sensitive to the size, one might hope that in this regime the expectation values would allow us to distinguish these various sources. Such hope is indeed realized, as we show in figure 9. In figures 9*a* and 9*c* we plot the extensions of the curves in figures 3*b* and 3*c*, respectively, up to $l \sim 6000$ (same sources and same conventions as used in figure 3 are used here as well). From figure 9*c*, we clearly see that we can distinguish the various sources for large enough l. All of these curves exhibit the large l exponential behavior characteristic of the spherically symmetric sources, as shown in figures 9*b* and 9*d*, which are just the respective logarithmic plots of figures 9*a* and 9*c*.

4. Generalization to $AdS_p \times S^q$

Above, we have encountered a special cancellation of the leading size signature for a spherical source in $AdS_5 \times S^5$. One may wonder whether this is a result of the fact that the curvature of the AdS and sphere have the same magnitude but opposite sign.

An affirmative answer would imply that our results could be drastically different in other spacetimes of interest for the AdS/CFT correspondence, namely in $AdS_4 \times S^7$ and in $AdS_7 \times S^4$, where the radii of curvature are different for the AdS and the sphere. To check this, we now extend the previous calculation to the more general case of $AdS_p \times S^q$.

The spacetime is described by the metric

$$ds^2 = -(\rho^2+1)\,dt^2 + \frac{d\rho^2}{\rho^2+1} + \rho^2\,d\Omega_{p-2}^2 + \alpha^2\left(d\chi^2 + \sin^2\chi\,d\Omega_{q-1}^2\right). \qquad (4.1)$$

As in the $AdS_5 \times S^5$ case, we set the radius of curvature of AdS to unity, so that α, which can be written as $\frac{q-1}{p-1}$, gives the radius of the sphere. The proper distance along the sphere direction now depends on α, so that (3.8) becomes

$$\hat{\rho} = \sinh^{-1}\rho\,, \qquad \hat{\chi} = \alpha\,\chi\,. \qquad (4.2)$$

The solution to the radial equation which is one at the origin is

$$\begin{aligned} R_l(\rho) &= F\left(-\frac{\lambda}{2}, \frac{\lambda}{2} + \frac{p-1}{2}, \frac{p-1}{2}; -\rho^2\right) \\ &\approx 1 + \frac{\lambda(\lambda+p-1)}{2(p-1)}\,\rho^2 + \frac{\lambda(\lambda-2)(\lambda+p-1)(\lambda+p+1)}{8(p^2-1)}\,\rho^4 + \cdots, \end{aligned} \qquad (4.3)$$

where $\lambda \equiv l/\alpha = l(\frac{p-1}{q-1})$. The spherical harmonic is given by $Y_l(\chi) \propto F(-l/2, l/2 + (q-1)/2, 1/2; \cos^2\chi)$ for even l, and $Y_l(\chi) \propto \cos\chi\, F((1-l)/2, (l+q)/2, 3/2; \cos^2\chi)$ for odd l. The small-χ expansion of the spherical harmonic (normalized to 1 at the north pole) is

$$Z_l(\chi) \approx 1 - \frac{l(l+q-1)}{2q}\chi^2 + \frac{l(l+q-1)\left(l^2+(q-1)l-\frac{2}{3}(q-1)\right)}{8q(q+2)}\chi^4 + \cdots. \qquad (4.4)$$

Finally, the volume form is modified, so that for a bi-spherical source on this spacetime, the coefficient of the asymptotic field becomes[6]

$$M_{l,\hat{0}} \propto \int_0^\pi \int_0^\infty s(\rho,\chi)\,R_l(\rho)\,Z_l(\chi)\,\rho^{p-2}\,\sin^{q-1}\chi\,d\rho\,d\chi\,. \qquad (4.5)$$

The small-l behavior of $M_{l,\hat{0}}$ for a uniform density spherical source $s(\rho,\chi) \propto \Theta(r_0^2 - (\sinh^{-1}\rho)^2 - \alpha^2\chi^2))$ is then given by

$$M_{l,\hat{0}} \propto 1 + \frac{13l(l+6)}{16128}r_0^4 + 0(r_0^6) \qquad (4.6)$$

for $AdS_4 \times S^7$, and

$$M_{l,\hat{0}} \propto 1 + \frac{5l(l+3)}{126}r_0^4 + 0(r_0^6) \qquad (4.7)$$

for $AdS_7 \times S^4$. This shows that these cases are qualitatively similar to $AdS_5 \times S^5$. The size continues to appear at order r_0^4 and not at order r_0^2.

[6]Although we are not considering expectation values here (since there is no dilaton in these cases), the massless scalar field can be viewed as a toy model for the linearized supergravity fields.

5. Discussion

We have seen that the expectation value of local operators in the CFT contain a considerable amount of information about small sources inside $AdS_5 \times S^5$. The most striking result is that, for a spherical source of radius r_0, the appropriately normalized expectation values of the operators $\mathcal{O}Z_l$ (defined after (2.20)) grow exponentially with lr_0^2 (3.18). So the size of a small source has a big effect on the expectation values. This exponential growth is obtained at large l. For small l, the size dependence first arises in a term of order r_0^4 (3.10). As discussed in section 3, one might have expected the size dependence to be an even larger effect. Since the curvature is noticable at $O(r^2)$, it could first show up at order r_0^2 for small l, and grow like the exponential of lr_0 for large l. It is not yet clear what is special about the spacetimes $AdS_p \times S^q$ arising in the AdS/CFT correspondence to cause this extra cancellation. In this regard, it would be interesting to consider spacetimes with less supersymmetry, e.g., $AdS \times K$. While we still expect a small source to produce exponentially growing expectation values, it is not clear whether the rate will be governed by lr_0 or lr_0^2.

We have also found an unusual behavior of the large l expectation values for ellipsoidal sources. When the source is slightly elongated in the AdS directions, the expectation values again grow exponentially, but when the source is slightly elongated in the sphere directions, they are quickly damped to zero (figure 8). While there are many examples of different supergravity configurations with the same local expectation values e.g. [14, 15], it is surprising that very similar supergravity configurations can yield vastly different expectation values.

We have considered static sources at the center of AdS_5. The extension to time dependent sources remains to be investigated, and is likely to have interesting consequences. For example, consider a source which is slowly oscillating from being extended in the AdS_5 directions to being slightly extended in the S^5 directions. It would appear that the normalized expectation values of operators $\mathcal{O}Z_l$ with large l in the field theory must change rapidly from being exponentially large to almost zero in each oscillation. As a second example, suppose one gives the center of mass a small velocity. Even though the spacetime is locally approximately Poincare invariant, the gauge theory description depends crucially on whether the velocity is in the S^5 or AdS_5 directions. If it is in the S^5 direction, the scalars in the nonzero expectation values will change, but the dependence on the S^3 at infinity will be unaffected. If the velocity is in the AdS_5 direction, the source will oscillate following a geodesic. This will cause the dependence on the S^3 to change, but the scalars in the expectation value will remain unchanged. For the special case of a point source, the expectation value for the $l = 0$ mode was computed in [16, 17]. One finds that the radial position of the source is correllated with the size of the excitation in the gauge theory, in accord with the UV/IR correspondence.

One application of our results is to the description of small black holes. With $AdS_5 \times S^5$ boundary conditions, it is possible for small ten dimensional black holes to be in static equilibrium with their Hawking radiation [18]. How would the CFT distinguish these states from other states with the same total energy? A state of pure radiation with this energy would not be localized on the S^5, and so would have zero expectation values for operators of the form $\langle OZ_l \rangle$. Since we do not yet know the exact supergravity solution corresponding to a small black hole in $AdS_5 \times S^5$, finding the field theory expectation values in this state remains an open problem. The results obtained here are not applicable due to nonlinear effects near the horizon. However, gravity is essentially linear for an object just ten times larger than a black hole, and we have seen that this would be reflected in the expectation values growing exponentially with l, with an exponent that depends on the radius of the star. So one could easily rule this out. Once one knows that there is a rather large mass contained within about ten Schwarzschild radii, the only possibility is a black hole. Although convincing, this argument is still indirect. We do not yet have a good description of the spacetime causal structure directly in the gauge theory. Note that one signal of an evaporating black hole is that the bound on the size of the source will decrease with time.

We remark that an alternate way to discern a black hole in the bulk through the local boundary operators would be by the use of probes. Namely, a suitably-designed probe thrown into a black hole would never reemerge, unlike the same probe thrown into a star or a thermal gas of radiation. Thus, after a sufficient time, we could learn about the bulk through the observed behavior of the probe. However, although this method would yield a more direct detection of the event horizon, and would still use just the local operators in the gauge theory, it requires following these operators for a time $\Delta t \sim \pi R$.

We have seen that asymptotically $AdS_5 \times S^5$ boundary conditions make a holographic description easier than for asymptotically flat spacetimes, since one can recover much more information about small objects from the asymptotic fields. Whether a purely holographic description exists for asymptotically flat spacetimes remains to be seen.

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